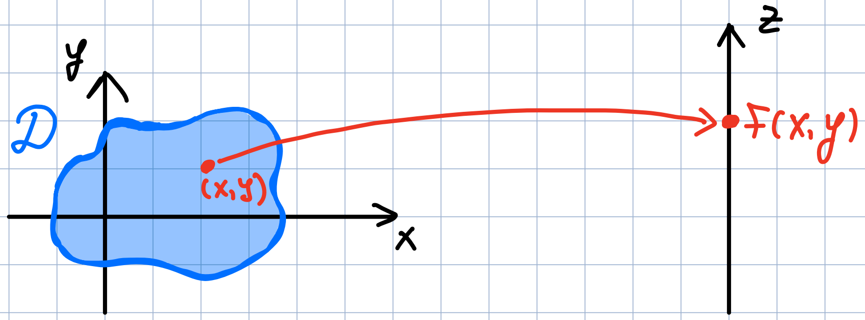
Last time: Functions of several variables

Function of two variables:

$f: (x,y) \longmapsto f(x,y)$

in $\underbrace{\mathcal{D}}_{\text{domain}} \subset \mathbb{R}^2$ in $\mathbb{R}$

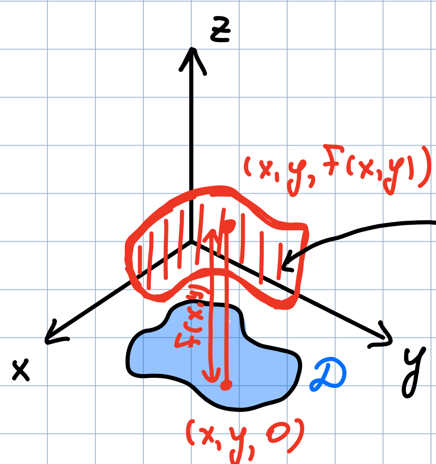


range: set of all values of $f(x,y) \subset \mathbb{R}$

domain: all (x,y) s.t. $f(x,y)$ is well-defined.

Graph:

Set of points (x,y,z) in $\mathbb{R}^3$ with $(x,y) \in \mathcal{D}$, $z = f(x,y)$

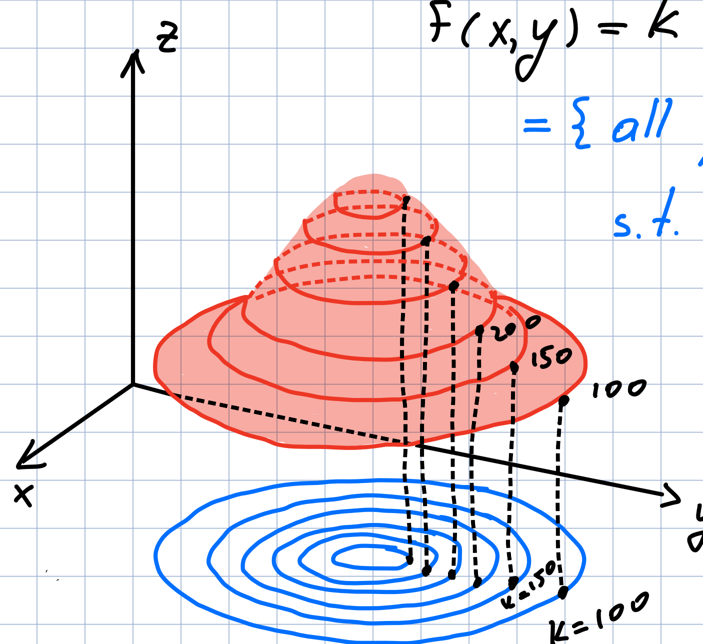


graph (surface) sitting above $\mathcal{D}$

Level curves:

curves with eq. $f(x,y) = k$, i.e.

$= \{$ all points (x,y) s.t. $f(x,y) = k \}$



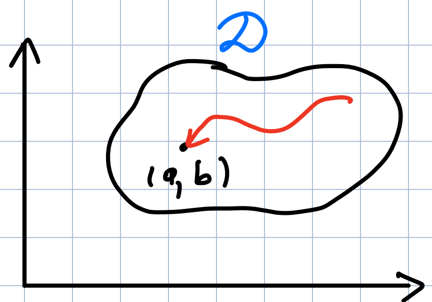
on this curve

Functions of 3 variables:

$f(x,y,z)$

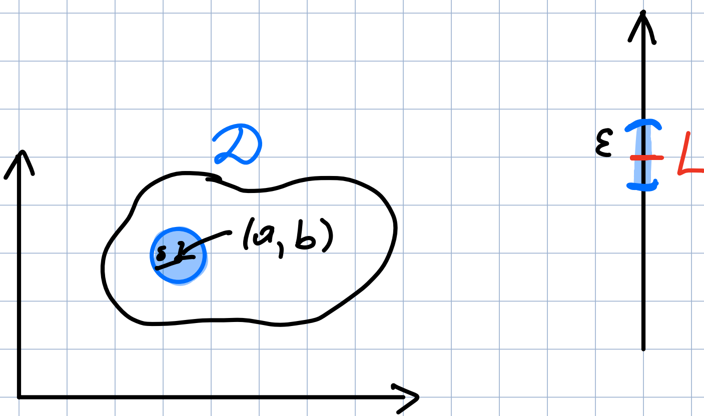
Limits and continuity:

$$\lim_{(x,y)\to(a,b)} f(x,y) = L$$ – The limit of $f(x,y)$ as (x,y) approaches (a,b)



• if values of $f(x,y)$ approach L as (x,y) approaches (a,b) along any path in the domain $\mathcal{D}$

• we can make $f(x,y)$ as close to L as we want by taking (x,y) sufficiently close to (a,b)



If for every number $\varepsilon > 0$ there is a corresponding number $\delta > 0$ s.t. if $(x,y) \in \mathcal{D}$ and $0 < \sqrt{(x-a)^2 + (y-b)^2} < \delta$ then $|f(x,y) - L| < \varepsilon$

Ex: $$f(x,y) = \frac{\sin(x^2+y^2)}{x^2+y^2}$$

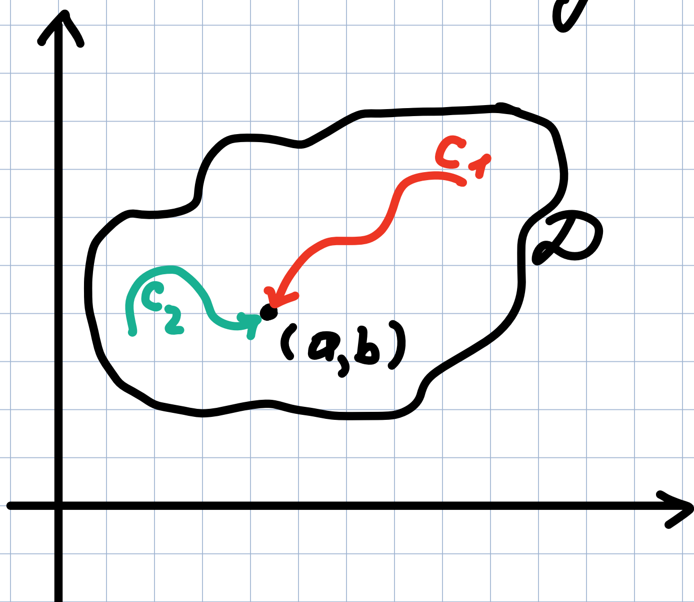
$$\lim_{(x,y)\to(0,0)} f(x,y) = 1$$

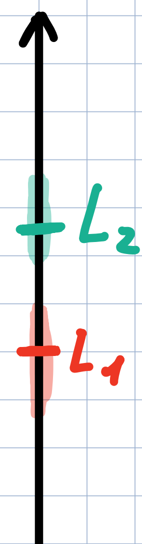
If $f(x,y) \to L_1$ as $(x,y) \to (a,b)$ along C_1

AND $f(x,y) \to L_2$ as $(x,y) \to (a,b)$ along C_2

where $L_1 \neq L_2$, then

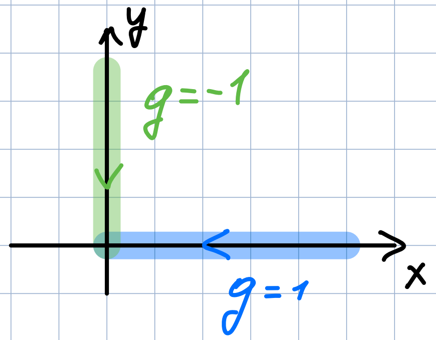
$$\lim_{(x,y)\to(a,b)} f(x,y) \text{ does NOT exist.}$$





Ex: $g(x,y) = \dfrac{x^2 - y^2}{x^2 + y^2}$ $\qquad \lim\limits_{(x,y)\to 0} g(x,y) = ?$

Sol:

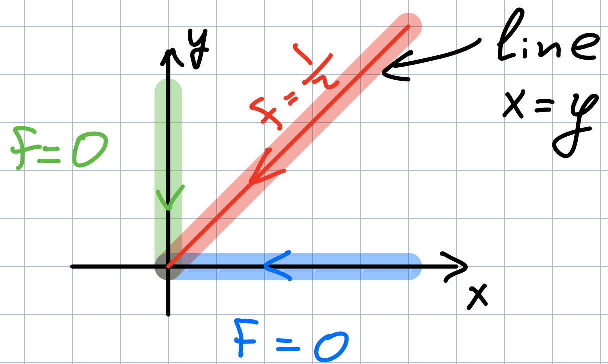


approaching along X-axis g stays 1

approaching along y-axis g stays -1

$\Rightarrow$ limit as $(x,y) \to (0,0)$ does NOT exist!

Ex: $f(x,y) = \dfrac{xy}{x^2 + y^2}$ $\qquad \lim\limits_{(x,y)\to 0} f(x,y) = ?$



approaching along X-axis f stays 0

approaching along y-axis f stays 0

BUT approaching along $x=y$ line $f = \frac{1}{2}$

$0 \neq \frac{1}{2}$

$\Rightarrow$ limit as $(x,y) \to (0,0)$ does NOT exist!

Ex: Find $\displaystyle\lim_{(x,y)\to(0,0)} \frac{3x^2y}{x^2+y^2}$ if it exists.

Sol: 1) We suspect the limit is 0.

2) Let $\varepsilon > 0$, WANTED $\delta > 0$ such that

if $0 < \sqrt{x^2+y^2} < \delta$ then $\left| \frac{3x^2y}{x^2+y^2} - 0 \right| < \varepsilon$

$\left(\Longleftrightarrow \frac{3x^2|y|}{x^2+y^2} < \varepsilon \right)$

3) Note that $x^2 \le x^2+y^2 \Rightarrow \frac{x^2}{x^2+y^2} \le 1$ therefore

$$\frac{3x^2|y|}{x^2+y^2} \le 3|y| = 3\sqrt{y^2} \le 3\sqrt{x^2+y^2} \le 3\delta$$

CHOOSE $\delta = \frac{\varepsilon}{3}$

$$\boxed{\lim_{(x,y)\to(0,0)} \frac{3x^2y}{x^2+y^2} = 0}$$

- $f(x,y)$ is continuous at (a,b) if $\lim_{(x,y)\to(a,b)} f(x,y) = f(a,b)$
- f is continuous on D if f is continuous at each point (a,b) in D

Ex: a polynomial: $f(x,y) = x^2 + xy + y^7$ is cont. on $\mathbb{R}^2$
a rational function: $g(x,y) = \dfrac{x+2y^2}{x^2+y^2}$ is cont. in its domain $\mathbb{R}^2 \setminus \{0,0\}$

Ex: Evaluate $\lim_{(x,y)\to(1,2)} (x^2y^3 - x^3y^2 + 3x + 2y)$

Sol: $f(x,y) = x^2y^3 - x^3y^2 + 3x + 2y$ is a polynomial $\Rightarrow$ cont. on $\mathbb{R}^2$

$$\Rightarrow \lim_{(x,y)\to(1,2)} f(x,y) = f(1,2) = 1^2 2^3 - 1^3 2^2 + 3(1) + 2(2) = 11$$

Ex: Where is $f(x,y) = \dfrac{x^2-y^2}{x^2+y^2}$ continuous?

Sol: Discontinuous at $(0,0)$ [NOT defined there]
Rational fct. is continuous on its domain $D = \{(x,y) \mid (x,y) \neq (0,0)\}$

Partial derivatives

$f(x,y)$ partial derivative w.r.t. x at (a,b):

set $g(x) = f(\underbrace{x}_{\text{variable}}, b)$ ← constant

← function of single variable

$$\boxed{f_x(a,b) = g'(a)}$$

likewise, partial derivative w.r.t. y at (a,b):

set $h(y) = f(a,y)$

$$\boxed{f_y(a,b) = h'(b)}$$

I.e. to find $f_x(x,y)$, view y as a constant, differentiate w.r.t. x

to find $f_y(x,y)$, view x as a constant, differentiate w.r.t. y

Notation: $\frac{\partial f}{\partial x} = f_x(x,y)$, $\frac{\partial f}{\partial y} = f_y(x,y)$

Ex: $f(x,y) = x^3 + x^2y^3 - 2y^2$ Find $f_x(2,1)$, $f_y(2,1)$.

Sol: $f_x(x,y) = 3x^2 + 2xy^3$ $f_x(2,1) = 3\cdot 2^2 + 2\cdot 2\cdot 1^3 = 16$

$f_y(x,y) = 3x^2y^2 - 4y$ $f_y(2,1) = 3\cdot 2^2\cdot 1^2 - 4\cdot 1 = 8$

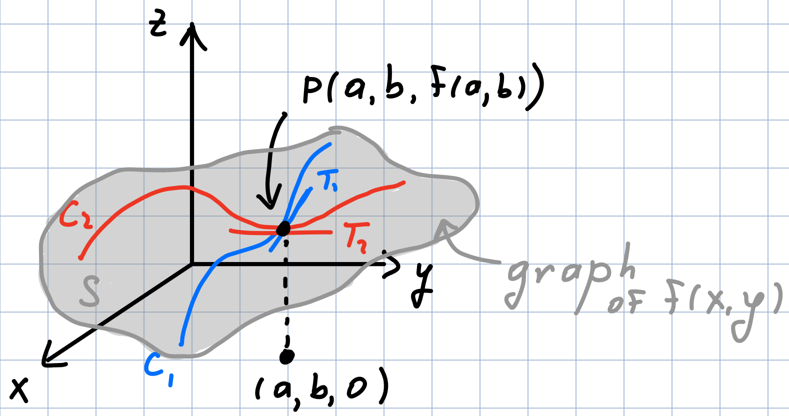
Ex: $f(x,y) = \sin\left(\frac{x}{1+y}\right)$ Find f_x and f_y

$$\frac{\partial f}{\partial x} = \cos\left(\frac{x}{1+y}\right)\cdot\frac{\partial}{\partial x}\left(\frac{x}{1+y}\right) = \cos\left(\frac{x}{1+y}\right)\cdot\frac{1}{1+y}$$

chain rule

$$\frac{\partial f}{\partial y} = \cos\left(\frac{x}{1+y}\right)\cdot\frac{\partial}{\partial y}\left(\frac{x}{1+y}\right) = -\cos\left(\frac{x}{1+y}\right)\frac{x}{(1+y)^2}$$

Interpretations of partial derivatives:



C_1: graph of $g(x) = f(x,b)$

= intersection of S with plane $y=b$

C_2: graph of $h(y) = f(a,y)$

= intersection of S with plane $x=a$

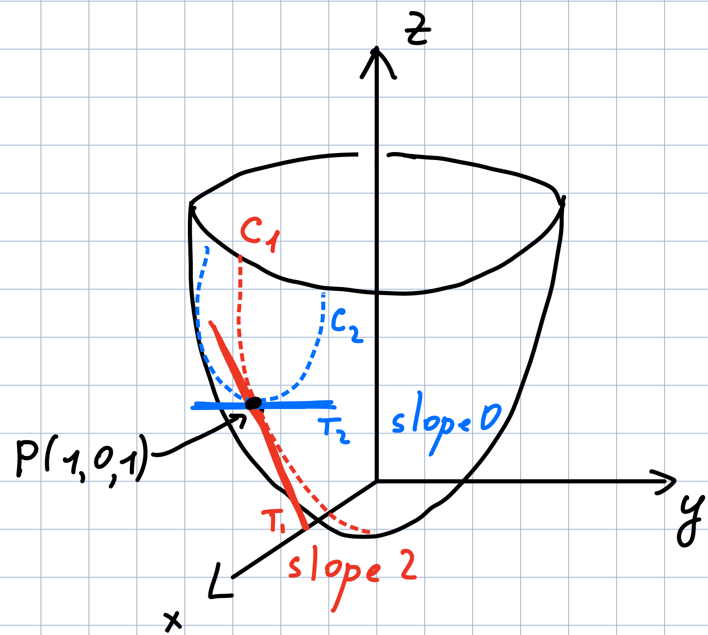
$f_x(a,b)$ = slope of tangent line T_1 to C_1 at $P(a,b,f(a,b))$

$f_y(a,b)$ = slope of tangent line T_2 to C_2 at $P(a,b,f(a,b))$

Ex: $f(x,y) = x^2 + y^2$

$f_x(1,0) = 2x\big|_{(1,0)} = 2$

$f_y(1,0) = 2y\big|_{(1,0)} = 0$



Higher derivatives:

$$f_{xx} = (f_x)_x = \frac{\partial^2 f}{\partial x^2}, \quad f_{xy} = (f_x)_y = \frac{\partial^2 f}{\partial x \partial y},$$
$$f_{yx} = (f_y)_x = \frac{\partial^2 f}{\partial y \partial x}, \quad f_{yy} = (f_y)_y = \frac{\partial^2 f}{\partial y^2}$$

2^{nd} order derivatives

Ex: $f(x,y) = x^3 + x^2 y^3 - 2y^2$

$f_x = 3x^2 + 2xy^3 \rightsquigarrow f_{xx} = 6x + 2y^3 \qquad f_{yx} = \underline{6xy^2}$

$f_y = 3x^2y^2 - 4y \qquad f_{xy} = \underline{6xy^2} \qquad f_{yy} = 6x^2y - 4$

Rmk: $f_{xy} = f_{yx}$ – this is true for any f!
– doesn't matter, in which order we are taking partial derivatives

Clairaut's Thm: $f_{xy}(a,b) = f_{yx}(a,b)$

// $f_{xyy} = f_{yxy} = f_{yyx}$ //

Ex: let z be defined implicitly by $x^3+y^3+z^3+6xyz=1$ $(*)$.

Find $\frac{\partial z}{\partial x}$, $\frac{\partial z}{\partial y}$.

Sol: apply $\frac{\partial}{\partial x}$ to $(*)$, keeping y constant:

$$3x^2 + 3z^2\frac{\partial z}{\partial x} + 6yz + 6xy\frac{\partial z}{\partial x} = 0$$

$$\Rightarrow \frac{\partial z}{\partial x} = -\frac{3x^2+6yz}{3z^2+6xy} = -\frac{x^2+2yz}{z^2+2xy}$$

apply $\frac{\partial}{\partial y}$ to $(*)$, keeping x constant:

$$3y^2 + 3z^2\frac{\partial z}{\partial y} + 6xz + 6xy\frac{\partial z}{\partial y} = 0$$

$$\Rightarrow \frac{\partial z}{\partial y} = -\frac{y^2+2xz}{z^2+2xy}$$

Ex: $f(x,y,z) = \sin(3x + yz)$. Calculate $f_{xxyz} = \frac{\partial^4 f}{\partial x^2 \partial y \partial z}$

Sol: $f_x = 3\cos(3x + yz)$

$f_{xx} = -9\sin(3x + yz)$

$f_{xxy} = -9z\cos(3x + yz)$

$f_{xxyz} = 9yz\sin(3x + yz) - 9\cos(3x + yz)$

Laplace equation: $\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0 \qquad (*)$

$\nwarrow$ partial differential equation

Ex: Check that $u(x,y) = e^x \sin y$ is a solution of $(*)$.

Sol: $u_x = e^x \sin y \qquad u_{xx} = e^x \sin y$

$u_y = e^x \cos y \qquad u_{yy} = -e^x \sin y$

$\left.\begin{matrix} \\ \end{matrix}\right\} \Rightarrow u_{xx} + u_{yy} = 0$

The Chain Rule:

Reminder: $y = f(x)$, $x = g(t)$ then y is inderectly a function of t

and $\frac{dy}{dt} = \frac{dy}{dx} \cdot \frac{dx}{dt} = f'(x) \cdot g'(t) = f'(g(t)) \cdot g'(t)$ [Case 0]

Case 1: $z = f(x, y)$, $x = g(t)$, $y = h(t)$

then $$\frac{dz}{dt} = \frac{\partial f}{\partial x} \cdot \frac{dx}{dt} + \frac{\partial f}{\partial y} \cdot \frac{dy}{dt}$$

$$\left(\text{or } \frac{\partial z}{\partial x} \cdot \frac{dx}{dt} + \frac{\partial z}{\partial y} \cdot \frac{dy}{dt}\right)$$

Ex: $z = \sqrt{1 + xy}$, $x = \sin t$, $y = t^2$. a) Find $\frac{dz}{dt}$ b) Find $\frac{dz}{dt}$ at $t = \pi$

Sol: (a) $$\frac{dz}{dt} = \frac{\partial z}{\partial x} \cdot \frac{dx}{dt} + \frac{\partial z}{\partial y} \cdot \frac{dy}{dt} = \frac{y}{2\sqrt{1+xy}} \cos t + \frac{x}{2\sqrt{1+xy}} 2t$$

$$= \frac{\frac{t^2}{2} \cos t + t \sin t}{\sqrt{1 + t^2 \sin t}}$$

(b) $t = \pi \Rightarrow x = 0, y = \pi^2 \rightsquigarrow -\frac{\pi^2}{2}$

Case 2: $z = f(x,y)$ and $x = g(s,t)$, $y = h(s,t)$

then $$\frac{\partial z}{\partial s} = \frac{\partial z}{\partial x}\cdot\frac{\partial x}{\partial s} + \frac{\partial z}{\partial y}\cdot\frac{\partial y}{\partial s}$$ and $$\frac{\partial z}{\partial t} = \frac{\partial z}{\partial x}\cdot\frac{\partial x}{\partial t} + \frac{\partial z}{\partial y}\cdot\frac{\partial y}{\partial t}$$

Ex: $z = (x-y)^5$, $x = s^2 t$, $y = st^2$. Find $\frac{\partial z}{\partial s}$, $\frac{\partial z}{\partial t}$.

Sol:
$$\frac{\partial z}{\partial s} = \frac{\partial z}{\partial x}\cdot\frac{\partial x}{\partial s} + \frac{\partial z}{\partial y}\cdot\frac{\partial y}{\partial s} = 5(x-y)^4 2st - 5(x-y)^4 t^2$$
$$= 5(s^2t - st^2)^4(2st - t^2)$$

$$\frac{\partial z}{\partial t} = \frac{\partial z}{\partial x}\cdot\frac{\partial x}{\partial t} + \frac{\partial z}{\partial y}\cdot\frac{\partial y}{\partial t} = 5(x-y)^4 s^2 - 5(x-y)^4 2st$$
$$= 5(s^2t - st^2)^4(s^2 - 2st)$$

General case of Chain Rule:

$$u = u(\underbrace{x_1, \dots, x_n}_{n \text{ variables}}) \quad \text{and for each } j: \quad x_j = x_j(\underbrace{t_1, \dots, t_m}_{m \text{ variables}})$$

then

$$\frac{\partial u}{\partial t_i} = \underbrace{\frac{\partial u}{\partial x_1}\frac{\partial x_1}{\partial t_i} + \dots + \frac{\partial u}{\partial x_n}\frac{\partial x_n}{\partial t_i}}_{n \text{ terms}}$$

Ex: $u = x + yz$, $\quad x = t$, $y = \ln t$, $z = \sin t$

$$\frac{du}{dt} = \frac{\partial u}{\partial x}\frac{dx}{dt} + \frac{\partial u}{\partial y}\frac{dy}{dt} + \frac{\partial u}{\partial z}\frac{dz}{dt} = 1 \cdot 1 + z\frac{1}{t} + y\cos t$$

$$= 1 + \frac{\sin t}{t} + \ln t \cos t$$

Implicit differentiation: If $y = f(x)$ is defined implicitly,
by $F(x, \underbrace{y}_{=f(x)}) = 0$

We can find $\frac{dy}{dx}$ via $0 = \frac{d}{dx} F(x,y) = \frac{\partial F}{\partial x} \underbrace{\frac{dx}{dx}}_{=1} + \frac{\partial F}{\partial y} \frac{dy}{dx}$

$$\Rightarrow \frac{dy}{dx} = - \frac{\partial F / \partial x}{\partial F / \partial y} = - \frac{F_x}{F_y}$$

Ex: y defined by $x^2 + y^3 = 1$. Find $y'(x)$.

Sol: $F(x,y) = x^2 + y^3 - 1 \quad \Rightarrow \quad \frac{dy}{dx} = - \frac{F_x}{F_y} = - \frac{2x}{3y^2}$

Ex: $z = z(x,y)$ defined by $x^2 + y^5 + z^3 = 0^{(*)}$. Find $\frac{\partial z}{\partial x}, \frac{\partial z}{\partial y}$

Sol: $\frac{\partial}{\partial x} (*)$: $2x + 0 + 3z^2 \frac{\partial z}{\partial x} = 0 \quad \Rightarrow \quad \frac{\partial z}{\partial x} = - \frac{2x}{3z^2}$

$\frac{\partial}{\partial y} (*)$: $0 + 5y^4 + 3z^2 \frac{\partial z}{\partial y} = 0 \quad \Rightarrow \quad \frac{\partial z}{\partial y} = - \frac{5y^4}{3z^2}$